

# TRIPLETS OF PURE FREE SQUAREFREE COMPLEXES

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**ABSTRACT.** On the category of bounded complexes of finitely generated free squarefree modules over the polynomial ring  $S$ , there is the standard duality functor  $\mathbb{D} = \text{Hom}_S(-, \omega_S)$  and the Alexander duality functor  $\mathbb{A}$ . The composition  $\mathbb{A} \circ \mathbb{D}$  is an endofunctor on this category, of order three up to translation. We consider complexes  $F^\bullet$  of free squarefree modules such that both  $F^\bullet, \mathbb{A} \circ \mathbb{D}(F^\bullet)$  and  $(\mathbb{A} \circ \mathbb{D})^2(F^\bullet)$  are pure, when considered as singly graded complexes. We conjecture i) the existence of such triplets of complexes for given triplets of degree sequences, and ii) the uniqueness of their Betti numbers, up to scalar multiple. We show that this uniqueness follows from the existence, and we construct such triplets if two of them are linear.

## INTRODUCTION

**Pure free resolutions** are free resolutions over the polynomial ring  $S$  of the form

$$S(-d_0)^{\beta_0} \leftarrow S(-d_1)^{\beta_1} \leftarrow \cdots \leftarrow S(-d_r)^{\beta_r}.$$

Their Betti diagrams have proven to be of fundamental importance in the study of Betti diagrams of graded modules over the polynomial ring. Their significance were put to light by the Boij-Söderberg conjectures, [2]. The existence of pure resolutions were first proven by D.Eisenbud, the author, and J.Weyman in [7] in characteristic zero, and by Eisenbud and F.-O.Schreyer in all characteristics, [8]. Later the methods of [8] were made more explicit and put into a larger framework, called tensor complexes, by C.Berkesch et.al. [1].

The Boij-Söderberg conjectures, settled in full generality in [8], concerns the stability theory of *Betti diagrams* of graded modules, i.e. it describes such diagrams up to multiplication by a positive rational number, or alternatively the positive rational cone generated by such diagrams. The Betti diagrams of pure resolutions are exactly the extremal rays in this cone. Two introductory papers on this theory are [11] and [9].

**Homological invariants.** The Betti diagram is however only part of the story when it comes to homological invariants of graded modules. A complex  $F^\bullet$  of free modules over the polynomial ring  $S$ , for instance a free resolution, comes with three sets of numerical homological invariants:

- $B$ : The graded Betti numbers  $\{\beta_{ij}\}$ ,
- $H$ : The Hilbert functions of the homology modules  $H^i(F^\bullet)$ ,

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- $C$ : The Hilbert functions of the cohomology modules. These modules are the homology modules of the dual complex  $\text{Hom}_S(F^\bullet, \omega_S)$ , where  $\omega_S$  is the canonical module.

It is then natural to approach the stability theory of the triplet data set  $(B, H, C)$ : Up to rational multiple, what triplets of such can occur? The recent article [6] has partial results in this direction. It describes the Betti diagrams of complexes  $F^\bullet$  with specified nondecreasing codimensions of the homology modules. We do not here investigate the question above directly, but we believe the following will be of relevance.

**Squarefree modules.** The notion of pure resolution or pure complex, has a very natural extension into *triplets of pure complexes*, in the setting of squarefree modules over the polynomial ring. Squarefree modules are  $\mathbb{N}^n$ -graded modules over the polynomial ring  $S = \mathbb{k}[x_1, \dots, x_n]$  and form a module category including squarefree monomial ideals, and Stanley-Reisner rings. Both the category of singly graded  $S$ -modules as well as squarefree  $S$ -modules, have the standard duality functor  $\mathbb{D} = \text{Hom}_S(-, \omega_S)$ . However for squarefree modules there is also another duality functor, Alexander duality  $\mathbb{A}$ . The composition  $\mathbb{A} \circ \mathbb{D}$  becomes an endofunctor on the category of bounded complexes of finitely generated free squarefree  $S$ -modules. (This is in fact the Auslander-Reiten translate on the derived category of complexes of squarefree modules, see [3].) There are two amazing facts concerning this endofunctor.

- The third iterate  $(\mathbb{A} \circ \mathbb{D})^3$  is isomorphic to the  $n$ 'th iterate of the translation functor on complexes, a result of K.Yanagawa, [19].
- The composition functor cyclically rotates the homological invariants: If  $F^\bullet$  has homological invariants  $(B, H, C)$  then
  - $\mathbb{A} \circ \mathbb{D}(F^\bullet)$  has homological invariants  $(H, C, B)$ , and
  - $(\mathbb{A} \circ \mathbb{D})^2(F^\bullet)$  has homological invariants  $(C, B, H)$ .

This is also implicit in [19].

Thus the various homology modules of  $F^\bullet$  are transferred to the various linear strands of  $\mathbb{A} \circ \mathbb{D}(F^\bullet)$  and the cohomology modules of  $F^\bullet$  are transferred to the linear strands of  $(\mathbb{A} \circ \mathbb{D})^2(F^\bullet)$ .

**The main idea** of this paper is to consider complexes  $F^\bullet$  of free squarefree modules such that (when considered as singly graded modules)

- $F^\bullet$  is pure,
- $\mathbb{A} \circ \mathbb{D}(F^\bullet)$  is pure,
- $(\mathbb{A} \circ \mathbb{D})^2(F^\bullet)$  is pure.

We call this a *triplet of pure complexes*. That  $F^\bullet$  is a pure resolution of a Cohen-Macaulay squarefree module, the classical case, corresponds to

- $F^\bullet$  is pure,
- $\mathbb{A} \circ \mathbb{D}(F^\bullet)$  is linear,
- $(\mathbb{A} \circ \mathbb{D})^2(F^\bullet)$  is linear.

**Construction of triplets.** Squarefree complexes are  $\mathbb{Z}^n$ -graded, or equivalently they are equivariant for the action of the diagonal matrices of  $GL(n)$ . That pure resolutions come with various group actions is the rule in the various constructions we have, [7], [1]. S.Sam and J.Weyman pursue this [16] in the context of other

linear algebraic groups. However being squarefree is something more than being  $\mathbb{Z}^n$ -graded. In particular for a squarefree complex  $F^\bullet$  it may happen that the only multidegree  $\mathbf{b}$  such that  $F^\bullet(-\mathbf{b})$  is squarefree, is the zero degree. It is therefore a priori not clear, even in the classical case, how to construct such complexes  $F^\bullet$ . As it turns out the tensor complexes of [1] make the perfect input for a construction, see in particular Remark 4.6. These tensor complexes are over a large polynomial ring  $S(V \otimes W_0^* \otimes \cdots \otimes W_{r+1}^*)$ . Letting  $V$  be the linear space  $\langle x_1, \dots, x_n \rangle$  and taking a general map

$$V \otimes W_0^* \otimes \cdots \otimes W_{r+1}^* \rightarrow V,$$

equivariant for the diagonal matrices in  $GL(n)$ , we may construct all cases of complexes  $F^\bullet$  corresponding to the classical case, Theorem 4.8. The existence of triplets of pure complexes in full generality, we state as Conjecture 2.10. In a subsequent paper, [12], we transfer this to a conjecture on the existence of certain complexes of coherent sheaves on projective spaces.

**Uniqueness of Betti numbers.** In the classical case the singly graded Betti numbers of  $F^\bullet$  (and also of the linear complexes  $\mathbb{A} \circ \mathbb{D}(F^\bullet)$  and  $(\mathbb{A} \circ \mathbb{D})^2(F^\bullet)$ ) are uniquely determined up to scalar multiple, by the degree sequence of  $F^\bullet$ .

It now turns out that for a triplet of pure complexes, given the degree sequences of each of the three complexes, the Betti numbers fulfill a number of homogeneous linear equations which is one less than the number of variables, i.e. the number of Betti numbers. We thus expect there to be a unique solution up to common rational multiple. Under the assumption that triplets of pure complexes exists (for all triplet of degree sequences fulfilling a simple necessary criterion), we show that the Betti numbers are uniquely determined up to common rational multiple, Theorem 3.9

**Pure resolutions in the squarefree setting** have previously been considered by W.Bruns and T.Hibi for Stanley-Reisner rings. In [4] they describe all possible degree sequences  $0 = d_0, d_1, \dots$  for pure resolutions of Stanley-Reisner rings with  $d_1 = 2$ , and classify the simplicial complexes where this occurs. When  $d_1 = 3$  they give a thorough investigation of possible degree sequences and the possible simplicial complexes, as well as interesting examples when  $d_1 \geq 4$ . They also give a complete classification of simplicial complexes where  $d_1 = m$  and  $d_2 = m - 1$  for  $m \geq 2$ . In [5] they classify Cohen-Macaulay posets where the Stanley-Reisner ring of the order complex has pure resolution. In [10] the author considers *Cohen-Macaulay designs* which in the language of the present article correspond to Cohen-Macaulay Stanley-Reisner rings with pure resolution and exactly three linear strands (so the Stanley-Reisner ideal has exactly two linear strands). Examples of such are cyclic polytopes and Alexander duals of Steiner systems.

However from the perspective of the present article, approaches in those directions are severely hampered by the fact that only for few degree sequences, by simple numerical considerations, can one hope that the first Betti number  $\beta_0$  may be chosen to be 1. For degree sequences where this value may be achieved these articles also testify to the difficulty in constructing pure resolutions of Stanley-Reisner rings. Our construction avoids the restriction  $\beta_0 = 1$ , rather making  $\beta_0$  large.

**Organization of article.** In Section 1 we give the setting of squarefree modules and the functors  $\mathbb{A}$  and  $\mathbb{D}$ . We show that they rotate the homological invariants of squarefree complexes. In Section 2 we develop the basic theory of triplets of pure complexes. We find a basic necessary condition, the balancing condition, on

the triplet of degree sequences of such complexes. We conjecture the existence of triplets of pure complexes for all balanced triplets of degree sequences, and the uniqueness of their Betti numbers, up to common scalar multiple, Conjecture 2.10. In Section 3 we show this uniqueness of Betti numbers, under the assumption that triplets of pure complexes do exist. In Section 4 we use the tensor complexes of [1] to construct triplets of pure complexes  $F^\bullet, \mathbb{A} \circ \mathbb{D}(F^\bullet)$  and  $(\mathbb{A} \circ \mathbb{D})^2(F^\bullet)$  when the last two complexes are linear.

## 1. DUALITY FUNCTORS AND ROTATION OF HOMOLOGICAL INVARIANTS

In this section we recall the notion of a squarefree module over the polynomial ring  $S = \mathbb{k}[x_1, \dots, x_n]$ , and the two duality functors we may define on the category of complexes of such modules, standard duality  $\mathbb{D}$  and Alexander duality  $\mathbb{A}$ .

A striking result of K.Yanagawa [19], says that the composition  $(\mathbb{A} \circ \mathbb{D})^3$  is naturally equivalent to the  $n$ 'th iterate of the translation functor on the derived category of squarefree modules. A complex of squarefree modules comes with three sets of homological invariants, the multigraded homology and cohomology modules, and the multigraded Betti spaces. We show that  $\mathbb{A} \circ \mathbb{D}$  cyclically rotates these invariants (which is a rather well known fact to experts).

**1.1. Squarefree modules and dualities.** Let  $S$  be the polynomial ring  $\mathbb{k}[x_1, \dots, x_n]$  where  $\mathbb{k}$  is a field. Let  $\epsilon_i$  be the  $i$ 'th coordinate vector in  $\mathbb{N}^n$ . An  $\mathbb{N}^n$ -graded  $S$ -module is called *squarefree*, introduced by K.Yanagawa in [18], if  $M$  is finitely generated and the multiplication map  $M_{\mathbf{b}} \xrightarrow{\cdot x_i} M_{\mathbf{b}+\epsilon_i}$  is an isomorphism of vector spaces whenever the  $i$ 'th coordinate  $b_i \geq 1$ . We denote the category of finitely generated squarefree  $S$ -modules by  $\text{sq-}S$ .

There is a one-one correspondence between subsets  $R \subseteq [n] = \{1, 2, \dots, n\}$  and multidegrees  $\mathbf{r}$  in  $\{0, 1\}^n$ , by letting  $R$  be the set of coordinates of  $\mathbf{r}$  equal to 1. By abuse of notation we shall often write  $R$  when strictly speaking we mean  $\mathbf{r}$ . For instance the degree  $\mathbf{r}$  part of  $M$ , which is,  $M_{\mathbf{r}}$  may be written  $M_R$ . Also if  $R$  is a set we shall if no confusion arises, denote its cardinality by the smaller case letter  $r$ . We also denote  $(1, 1, \dots, 1)$  as  $\mathbf{1}$ . Take note that a squarefree module is completely determined, up to isomorphism, by the graded pieces  $M_R$  and the multiplication maps between them

$$M_R \xrightarrow{\cdot x_v} M_{R \cup \{v\}}$$

where  $v \notin R$ .

If  $M$  is a squarefree module and  $0 \leq d \leq n$ , its *squarefree part of degree  $d$*  is

$$\bigoplus_{|R|=d} M_R.$$

Note that taking squarefree parts is an exact functor from squarefree modules to vector spaces. In particular note that the squarefree part of  $S(-\mathbf{b})$  in degree  $d$  has dimension  $\binom{n-|\mathbf{b}|}{d-|\mathbf{b}|} = \binom{n-|\mathbf{b}|}{n-d}$ .

For a squarefree module  $M$  there is a notion of Alexander dual module  $\mathbf{A}(M)$ , defined by T.Römer [15] and E.Miller [14]. For  $R$  a subset of  $[n]$ , let  $R^c$  be its complement. Then  $\mathbf{A}(M)_R$  is the dual  $\text{Hom}_{\mathbb{k}}(M_{R^c}, k)$ . If  $v$  is not in  $R$  the multiplication

$$\mathbf{A}(M)_R \xrightarrow{\cdot x_v} \mathbf{A}(M)_{R \cup \{v\}}$$

is the dual of the multiplication

$$M_{(R \cup \{v\})^c} \xrightarrow{\cdot x_v} M_{R^c}.$$

By obvious extension this defines  $\mathbf{A}(M)_{\mathbf{b}}$  for all  $\mathbf{b}$  in  $\mathbb{N}^n$  and all multiplications.

*Example 1.1.* If  $S = \mathbb{k}[x_1, x_2, x_3, x_4]$  then the Alexander dual of  $S(-(1, 0, 1, 1))$  is  $S/(x_1, x_3, x_4)$ . The Alexander dual of  $S(-\mathbf{1})$  is the simple quotient module  $k$ .

For a multidegree  $\mathbf{b}$  in  $\mathbb{Z}^n$ , the free  $S$ -module  $S(-\mathbf{b})$  is a squarefree module if and only if  $\mathbf{b} \in \{0, 1\}^n$ , i.e. all coordinates of  $\mathbf{b}$  are 0 or 1. Direct sums of such modules are the *free squarefree  $S$ -modules*. Denote by  $\text{fsq-}S$  the category of finitely generated such modules.

Let  $C^b(\text{sq-}S)$  and  $C^b(\text{fsq-}S)$  be the categories of bounded complexes of finitely generated squarefree, resp. free squarefree modules. There is a natural duality

$$\mathbb{D} : C^b(\text{fsq-}S) \longrightarrow C^b(\text{fsq-}S)$$

defined by

$$\mathbb{D}(F^\bullet) = \text{Hom}_S(F^\bullet, S(-\mathbf{1})),$$

so in particular  $\mathbb{D}(S(-\mathbf{b})) = S(\mathbf{b}-\mathbf{1})$ . We would also like to define Alexander duality on the category  $C^b(\text{fsq-}S)$ . However there is a slight problem in that Alexander duality as defined above does not take free modules to free modules.

To remedy this, any bounded complex of squarefree modules  $X^\bullet$  has a minimal resolution  $F^\bullet \rightarrow X^\bullet$  by free squarefree modules. This defines a functor  $\text{res} : C^b(\text{sq-}S) \rightarrow C^b(\text{fsq-}S)$ . (There is of course also a natural inclusion  $\iota : C^b(\text{fsq-}S) \rightarrow C^b(\text{sq-}S)$ .) We now define Alexander duality

$$\mathbb{A} : C^b(\text{fsq-}S) \rightarrow C^b(\text{fsq-}S)$$

by letting  $\mathbb{A}$  be the composition  $\text{res} \circ \mathbf{A}$  where  $\mathbf{A}$  is the Alexander duality defined above.

*Example 1.2.* Continuing the example above, a free resolution of  $S/(x_1, x_2, x_4)$  is

$$\begin{array}{ccccccc} S & \longleftarrow & S^3 & \longleftarrow & S^3 & \longleftarrow & S \\ (0, 0, 0, 0) & & (1, 0, 0, 0) & & (1, 0, 1, 0) & & (1, 0, 1, 1) \\ & & (0, 0, 1, 0) & & (1, 0, 0, 1) & & \\ & & (0, 0, 0, 1) & & (0, 0, 1, 1) & & \end{array}$$

where we have written below the multidegrees of the generators. Then the Alexander dual  $\mathbb{A}(S(-(1, 0, 1, 1)))$  is the above resolution.

By composing with the resolution we may also consider  $\mathbb{A}$  and  $\mathbb{D}$  as functors on  $C^b(\text{sq-}S)$

$$C^b(\text{sq-}S) \xrightarrow{\text{res}} C^b(\text{fsq-}S) \xrightarrow{\mathbb{A}, \mathbb{D}} C^b(\text{fsq-}S).$$

For a complex  $X^\bullet$  let  $X^\bullet[-p]$  be its  $p$ 'th cohomological translate, i.e.  $(X^\bullet[-p])^q = X^{q-p}$ . Yanagawa, [19], shows that  $(\mathbb{A} \circ \mathbb{D})^3$  is isomorphic to the  $n$ 'th iterate  $[n]$  of the translation functor.

**1.2. Homological invariants.** The complex  $X^\bullet$  comes with three sets of squarefree homological invariants. First there is the homology

$$H_R^i(X^\bullet) := (H^i(X^\bullet))_R$$

where  $i \in \mathbb{Z}$  and  $R \subseteq [n]$ . For a vector space  $V$ , denote by  $V^*$  its dual  $\text{Hom}_{\mathbb{k}}(V, \mathbb{k})$ . We define the cohomology as

$$C_R^i(X^\bullet) := (H^{-i}(\mathbb{D}(X^\bullet))_{R^c})^*.$$

Note that by local duality, if  $X^\bullet$  is a module  $M$ , then this relates to local cohomology by

$$C_R^{i-n}(M) = H_{\mathbf{r}}^i(M)_{\mathbf{r}-1}$$

where  $\mathbf{r}$  is the 0, 1-vector with support  $R$ . Thirdly a minimal free squarefree resolution  $F^\bullet$  of  $X^\bullet$  has terms which may be written  $F_i = \bigoplus_{R \subseteq [n]} S \otimes_{\mathbb{k}} B_R^i$  and we define the Betti spaces to be

$$B_R^i(X^\bullet) := (\text{Tor}_i^S(X^\bullet, k)_R) = (B_R^i).$$

Now a basic and very interesting fact is that the functors  $\mathbb{A}$  and  $\mathbb{D}$  interchange the homology, cohomology, and Betti spaces. First we consider  $\mathbb{D}$ .

**Lemma 1.3.** *The functor  $\mathbb{D}$  interchanges the homological invariants of  $X^\bullet$  as follows.*

- $B_R^i(\mathbb{D}(X^\bullet)) = B_{R^c}^{-i}(X^\bullet)^*.$
- $H_R^i(\mathbb{D}(X^\bullet)) = C_{R^c}^{-i}(X^\bullet)^*.$
- $C_R^i(\mathbb{D}(X^\bullet)) = H_{R^c}^{-i}(X^\bullet)^*.$

*Proof.* This is clear. □

Before describing how the functor  $\mathbb{A}$  interchanges the homological invariants, we recall a basic fact from [19]. For a square-free module  $M$ , one may define a complex  $\mathcal{L}(M)$  (see [19, p.9] where it is denoted by  $\mathcal{F}(M)$ ) by

$$\mathcal{L}^i(M) = \bigoplus_{|R|=i} (M_R)^\circ \otimes_{\mathbb{k}} S$$

where  $(M_R)^\circ$  is  $M_R$  but considered to have multidegree  $R^c$ . The differential is

$$m^\circ \otimes s \mapsto \sum_{j \notin R} (-1)^{\alpha(j, R)} (x_j m)^\circ \otimes x_j s$$

where  $\alpha(j, R)$  is the number of  $i$  in  $R$  such that  $i < j$ .

For a minimal complex  $F^\bullet$  of free squarefree  $S$ -modules define its  $i$ 'th linear strand  $F_{\langle i \rangle}^\bullet$  to have terms

$$F_{\langle i \rangle}^j = \bigoplus_{|R|=i-j} S \otimes_{\mathbb{k}} B_R^j.$$

Since  $F^\bullet$  is minimal, the  $i$ 'th linear strand is naturally a complex. The following is [19, Thm. 3.8].

**Proposition 1.4.** *The  $i$ 'th linear strand of  $\mathbb{A} \circ \mathbb{D}(X^\bullet)$  is*

$$\mathcal{L}(H^i(X^\bullet))[n - i].$$

This gives the following.



The following will be of particular interest and motivation in the next Section 2.

**Lemma 1.9.** *The complexes  $\mathbb{A} \circ \mathbb{D}(F^\bullet)$  and  $(\mathbb{A} \circ \mathbb{D})^2(F^\bullet)$  are both linear if and only if  $F^\bullet$  is a resolution of a Cohen-Macaulay module.*

*Proof.* The various homology modules of  $F^\bullet$  are translated to the various linear strands of  $(\mathbb{A} \circ \mathbb{D})(F^\bullet)$ . So  $F^\bullet$  has only one nonzero homology module iff  $(\mathbb{A} \circ \mathbb{D})(F^\bullet)$  is linear. Similarly the cohomology of  $F^\bullet$  is translated to the Betti spaces of  $(\mathbb{A} \circ \mathbb{D})^2(F^\bullet)$  so  $F^\bullet$  has only one nonzero cohomology module iff  $(\mathbb{A} \circ \mathbb{D})^2(F^\bullet)$  is linear. But the fact that  $F^\bullet$  has only one nonzero homology module  $M$  and  $\mathbb{D}(F^\bullet)$  has only one nonzero homology module is equivalent to  $M$  being a Cohen-Macaulay module.  $\square$

**1.3. The functor  $\mathbb{A} \circ \mathbb{D}$  on a basic class of modules.** For  $A \subseteq [n]$ , the module  $S(-A)$  is a projective module. Denote by  $S/A = S/(x_i)_{i \in A}$ . (This is an injective module in  $\text{sq-}S$ .)

More generally for a partition  $A \cup B \cup C$  of  $[n]$ , the module  $(S/A)(-B)$  will be a squarefree module. Let us denote it as  $S/A(-B; C)$ . These form a basic simple class of squarefree modules closed with respect to the functors  $\mathbb{A}$  and  $\mathbb{D}$  when we identify modules with their minimal resolutions.

**Lemma 1.10.** *Let  $A \cup B \cup C$  be a partition of  $[n]$ .*

1. *There is a quasi-isomorphism*

$$\mathbb{D}(S/A(-B; C)) \xrightarrow{\cong} S/A(-C; B)[-a].$$

2. *There is a quasi-isomorphism*

$$\mathbb{A}(S/A(-B; C)) \xrightarrow{\cong} S/B(-A; C).$$

*Proof.* For  $A \subseteq [n]$  denote by  $\mathbb{k}A$  the vector space generated by  $x_i, i \in A$ . The projective resolution of  $S/A(-B)$  is

$$P^\bullet : S(-B) \leftarrow S(-B) \otimes (\mathbb{k}A) \leftarrow S(-B) \otimes \wedge^2(\mathbb{k}A) \leftarrow \cdots \leftarrow S(-B) \otimes \wedge^a(\mathbb{k}A).$$

The dual complex  $\text{Hom}_S(P^\bullet, S(-1))$  is  $\mathbb{D}(S/A(-B))$ . Since the last term  $S(-B) \otimes \wedge^a(\mathbb{k}A)$  in  $P^\bullet$  is generated in degree  $A \cup B$ , the dual complex is

$$S(-C) \leftarrow S(-C) \otimes (\mathbb{k}A) \leftarrow S(-C) \otimes \wedge^2(\mathbb{k}A) \leftarrow \cdots \leftarrow S(-C) \otimes (\mathbb{k}A)^a,$$

a resolution of  $S/A(-C; B)$ .

To see the second part of the lemma, it is not difficult to verify that the Alexander dual  $\mathbb{A}(S/A(-B; C)) \cong S/B(-A; C)$ .  $\square$

We then get the following diagram.

$$\begin{array}{ccccc}
 & & S/C(-B; A)[-a-c] & & \\
 & \nearrow \mathbb{D} & & \nwarrow \mathbb{A} & \\
 S/C(-A; B)[a] & & & & S/B(-C; A)[a+c] \\
 \uparrow \mathbb{A} & & & & \downarrow \mathbb{D} \\
 S/A(-C; B)[-a] & & & & S/B(-A; C)[-a-b-c] \\
 & \nwarrow \mathbb{D} & & \nearrow \mathbb{A}[-n] & \\
 & & S/A(-B; C) & & 
 \end{array}$$



A particular case is the following diagram.

$$\begin{array}{ccccc}
 & & \mathbb{k}[-n] & & \\
 & \nearrow \mathbb{D} & & \nwarrow \mathbb{A} & \\
 \mathbb{k} & & & & S(-1)[n] \\
 \uparrow \mathbb{A} & & & & \downarrow \mathbb{D} \\
 S(-1) & & & & S[-n] \\
 & \nwarrow \mathbb{D} & & \nearrow \mathbb{A}[-n] & \\
 & & S & & 
 \end{array}$$

## 2. TRIPLETS OF PURE COMPLEXES

As stated in the introduction the importance of pure free resolutions of Cohen-Macaulay  $S$ -modules is established with the Boij-Söderberg conjectures, and their subsequent demonstration.

A complex of free  $S$ -modules  $F^\bullet$  is *pure* if it has the form

$$F^\bullet : S(-d_0)^{\beta_0} \leftarrow S(-d_1)^{\beta_1} \leftarrow \cdots \leftarrow S(-d_r)^{\beta_r}$$

for some integers  $d_0 < d_1 < \cdots < d_r$ . These integers are the *degree sequence* of the pure complex.

We shall investigate the condition that *all three complexes*  $F^\bullet$ ,  $(\mathbb{A} \circ \mathbb{D})(F^\bullet)$  and  $(\mathbb{A} \circ \mathbb{D})^2(F^\bullet)$  are *pure* when considered as singly graded complexes. By Lemma 1.9 the special case that  $F^\bullet$  is a pure resolution of a Cohen-Macaulay module corresponds to the case that  $F^\bullet$  is pure while  $(\mathbb{A} \circ \mathbb{D})(F^\bullet)$  and  $(\mathbb{A} \circ \mathbb{D})^2(F^\bullet)$  are both linear complexes.

**2.1. Basic properties and examples.** We now give an example of a triplet of pure complexes, but let us first give a lemma telling how  $\mathbb{A} \circ \mathbb{D}(F^\bullet)$  may be computed.

**Lemma 2.1.** *Let*

$$F^\bullet : \cdots \rightarrow F^i \rightarrow F^{i+1} \rightarrow \cdots .$$

*Then  $\mathbb{A} \circ \mathbb{D}(F^\bullet)$  is homotopy equivalent to the total complex of*

$$\cdots \rightarrow \mathbb{A} \circ \mathbb{D}(F^i) \rightarrow \mathbb{A} \circ \mathbb{D}(F^{i+1}) \rightarrow \cdots .$$

*Proof.* Recall Alexander duality  $\mathbf{A}$  on the category of squarefree modules. The complex  $\mathbf{A} \circ \mathbb{D}(F^\bullet)$  is simply the complex of modules

$$\cdots \rightarrow \mathbf{A} \circ \mathbb{D}(F^i) \rightarrow \mathbf{A} \circ \mathbb{D}(F^{i+1}) \rightarrow \cdots .$$

Now if

$$(1) \quad \cdots \rightarrow M^i \rightarrow M^{i+1} \rightarrow \cdots$$

is a sequence of modules and  $F^{i,\bullet} \rightarrow M^i$  is a free resolution, we may lift the differentials  $M^i \rightarrow M^{i+1}$ , to differentials  $F^{i,\bullet} \rightarrow F^{i+1,\bullet}$ . Then the total complex of

$$\cdots \rightarrow F^{i,\bullet} \rightarrow F^{i+1,\bullet} \rightarrow \cdots$$

will be a resolution of (1) and hence it is homotopy equivalent to a minimal free resolution of this complex. Whence the result follows since  $\mathbb{A} \circ \mathbb{D}(F^i)$  is a free resolution of  $\mathbf{A} \circ \mathbb{D}(F^i)$ .  $\square$

*Example 2.2.* Let  $S = \mathbb{k}[x_1, x_2, x_3]$ . Consider the complex

$$F^\bullet : S \xleftarrow{[x_1x_2, x_1x_3, x_2x_3]} S(-2)^3.$$

First we find  $(\mathbb{A} \circ \mathbb{D})(F^\bullet)$ . By the figures of Subsection 1.3,  $\mathbb{A} \circ \mathbb{D}(S)$  is isomorphic to  $\mathbb{k}$ , and the resolution is the Koszul complex

$$S \leftarrow S(-1)^3 \leftarrow S(-2)^3 \leftarrow S(-3).$$

(It is really multigraded but for simplicity we only depict it as singly graded.) Also  $\mathbb{A} \circ \mathbb{D}(S(-([3] \setminus \{i\})))$  is isomorphic to  $S/(x_i)$  and so has resolution  $S \xleftarrow{x_i} S(-1)$ .

Therefore  $\mathbb{A} \circ \mathbb{D}(F^\bullet)$  is a minimal version of the total complex of

$$\begin{array}{ccccccc} S^3 & \longleftarrow & S(-1)^3 & & & & \\ \downarrow & & \downarrow & & & & \\ S & \longleftarrow & S(-1)^3 & \longleftarrow & S(-2)^3 & \longleftarrow & S(-3) \end{array}$$

It is easily seen that such minimal version is is

$$S^2 \xleftarrow{\begin{bmatrix} x_1x_2 & -x_1x_3 & 0 \\ 0 & x_1x_3 & -x_2x_3 \end{bmatrix}} S(-2)^3 \xleftarrow{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} S(-3).$$

Now consider  $(\mathbb{A} \circ \mathbb{D})^2(F^\bullet)$ . By Lemma 1.10,  $(\mathbb{A} \circ \mathbb{D})^2(S)$  is isomorphic to  $S(-3)$  and  $(\mathbb{A} \circ \mathbb{D})^2(S(-\{1, 2\}))$  is isomorphic to  $S/(x_1, x_2)(-\{3\})[-2]$ . Therefore  $(\mathbb{A} \circ \mathbb{D})^2(F^\bullet)$  is a minimal version of the total complex of

$$\begin{array}{ccc} S(-1)^3 & \longleftarrow & S(-2)^6 \longleftarrow S(-3)^3 \\ & & \downarrow \\ & & S(-3). \end{array}$$

Such a minimal version is then

$$S(-1)^3 \xleftarrow{\begin{bmatrix} x_1 & x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_2 & x_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_3 & x_1 \end{bmatrix}} S(-2)^6 \xleftarrow{\begin{bmatrix} x_2 & -x_1 & -x_3 & x_2 & 0 & 0 \\ 0 & 0 & x_3 & -x_2 & -x_1 & x_3 \end{bmatrix}^t} S(-3)^2.$$

In summary

$$\begin{aligned} F^\bullet &: S \leftarrow S(-2)^3 \\ \mathbb{A} \circ \mathbb{D}(F^\bullet) &: S^2 \leftarrow S(-2)^3 \leftarrow S(-3) \\ (\mathbb{A} \circ \mathbb{D})^2(F^\bullet) &: S(-1)^3 \leftarrow S(-2)^6 \leftarrow S(-3)^2. \end{aligned}$$

So all complexes are pure, and two of them are not linear.

**Lemma 2.3.** *Let*

$$F^\bullet : S(-a_0)^\alpha \leftarrow S(-a_1)^{\alpha'} \leftarrow \dots$$

*be a pure complex of squarefree modules with final term  $S(-a_0)^\alpha$  in cohomological position  $t$ . If  $\mathbb{A} \circ \mathbb{D}(F^\bullet)$  also is a pure complex, then*

$$\mathbb{A} \circ \mathbb{D}(F^\bullet) : \dots \leftarrow S(-n + a_0)^\alpha$$

*where the initial term  $S(-n + a_0)^\alpha$  is in cohomological position  $-n + a_0 + t$ .*

In particular, the initial terms of  $\mathbb{D}(F^\bullet)$  and its Alexander dual  $\mathbb{A} \circ \mathbb{D}(F^\bullet)$  are both equal to  $S(-n + a_0)^\alpha$ .

*Proof.* Considered as a complex of graded modules,  $\mathbb{A} \circ \mathbb{D}(F^\bullet)$  is the total complex of

$$\begin{array}{ccccccc} S^{\alpha'} & \longleftarrow & S(-1)^{\alpha'(n-a_1)} & \longleftarrow & \dots & & \\ \downarrow & & \downarrow & & & & \\ S^\alpha & \longleftarrow & S(-1)^{\alpha(n-a_0)} & \longleftarrow & \dots & \longleftarrow & S(-n + a_0)^\alpha \end{array}$$

When making a minimal complex of the total complex,  $S(-n + a_0)^\alpha$  cannot cancel out, so it must be the last term. Since  $S^\alpha$  is in cohomological position  $t$ , the last term must be in cohomological position  $-n + a_0 + t$ .  $\square$

In a pure complex

$$(2) \quad F^\bullet : S(-a_0)^{\alpha_0} \longleftarrow S(-a_1)^{\alpha_1} \longleftarrow \dots \longleftarrow S(-a_r)^{\alpha_r}$$

an integer  $d$  is called a *degree* of this complex if  $d = a_i$  for some  $i$ . Otherwise it is called a nondegree. If the nondegree is in  $[a_0, a_r]$  it is an *internal* nondegree.

Now suppose we have a situation where  $(\mathbb{A} \circ \mathbb{D})^i(F^\bullet)$  are pure complexes for  $i = 0, 1$  and 2. Write the complexes as:

$$\begin{aligned} F^\bullet & : S(-a_0)^{\alpha_0} \longleftarrow S(-a_1)^{\alpha_1} \longleftarrow \dots \longleftarrow S(-a_{r_0})^{\alpha_{r_0}} \\ \mathbb{A} \circ \mathbb{D}(F^\bullet) & : S(-b_0)^{\beta_0} \longleftarrow S(-b_1)^{\beta_1} \longleftarrow \dots \longleftarrow S(-b_{r_1})^{\beta_{r_1}} \\ (\mathbb{A} \circ \mathbb{D})^2(F^\bullet) & : S(-c_0)^{\gamma_0} \longleftarrow S(-c_1)^{\gamma_1} \longleftarrow \dots \longleftarrow S(-c_{r_2})^{\gamma_{r_2}}. \end{aligned}$$

We denote by  $A$  the set of degrees of  $F^\bullet$ , and similarly  $B$  and  $C$  for the degrees of  $\mathbb{A} \circ \mathbb{D}(F^\bullet)$  and  $(\mathbb{A} \circ \mathbb{D})^2(F^\bullet)$ . The triplet  $(A, B, C)$  is the *degree triplet* of the triplet of pure complexes. Let  $e_A$  be the number of internal nondegrees of  $F^\bullet$ , and correspondingly we define  $e_B$  and  $e_C$ . Let  $e$  be the total number of internal nondegrees for the triplet,  $e_A + e_B + e_C$ . As they turn out to be central invariants, we let  $c = a_0$ ,  $a = b_0$  and  $b = c_0$ .

**Proposition 2.4.** *a. The degrees in the last terms of the complexes above are  $a_0 = -n + b$ ,  $b_1 = -n + c$ , and  $c_2 = -n + a$ .*

*b. The number of variables  $n = a + b + c + e$ .*

*Proof.* Part a. is by Lemma 2.3 above. Also, by the lemma above, if  $S(-a_0)^{\alpha_0}$  in  $F^\bullet$  is in cohomological position  $t$ , then  $S(-b_{r_1})^{\beta_{r_1}}$  in  $\mathbb{A} \circ \mathbb{D}(F^\bullet)$  is in position  $t - b_{r_1}$ , and so the first term  $S(-b)^{\beta_0}$  is in position  $t - b_{r_1} + r_1$ . But  $r_1 + e_B = b_{r_1} - b_0$ , and so this position is  $t - a - e_B$ . Applying the lemma again, we get that  $S(-c)^{\gamma_0}$  in  $(\mathbb{A} \circ \mathbb{D})^2(F^\bullet)$  is in position  $t - a - b - e_B - e_C$ . And then again we get that  $S(-a)^{\alpha_0}$  in  $(\mathbb{A} \circ \mathbb{D})^3(F^\bullet)$  is in position  $t - a - b - c - e$ .

But since  $(\mathbb{A} \circ \mathbb{D})^3$  is isomorphic to the  $n$ 'th iterate of the translation functor, we get that  $n = a + b + c + e$ .  $\square$

We can represent the degrees of the complex  $F^\bullet$  as a string of circles indexed by the integers from  $a_0 = c$  to  $a_{r_0} = n - b$  by letting a circle be filled  $\bullet$  if it is at a position  $a_i$  and be a blank circle  $\circ$  otherwise.

*Example 2.5.* A complex

$$S(-1)^6 \leftarrow S(-3)^{27} \leftarrow S(-4)^{24} \leftarrow S(-7)^3$$

with  $n = 9$ , gives rise to the diagram

$$\overset{1}{\bullet} \leftarrow \overset{2}{\circ} \leftarrow \overset{3}{\bullet} \leftarrow \overset{4}{\bullet} \leftarrow \overset{5}{\circ} \leftarrow \overset{6}{\circ} \leftarrow \overset{7}{\bullet}.$$

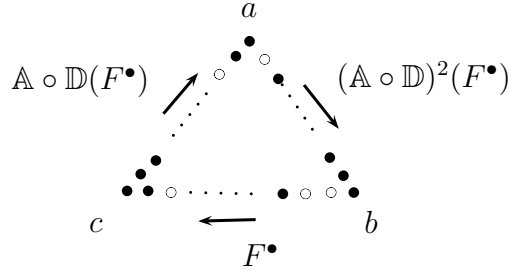
The dual complex  $\mathbb{D}(F^\bullet) = \text{Hom}_S(F^\bullet, S(-1))$ , which is

$$S(-8)^6 \rightarrow S(-6)^{27} \rightarrow S(-5)^{24} \rightarrow S(-2)^3,$$

gives a diagram by switching the orientation above and letting the numbering be

$$\overset{8}{\bullet} \rightarrow \overset{7}{\circ} \rightarrow \overset{6}{\bullet} \rightarrow \overset{5}{\bullet} \rightarrow \overset{4}{\circ} \rightarrow \overset{3}{\circ} \rightarrow \overset{2}{\bullet}.$$

All three complexes may be represented in a triangle, called the *degree triangle* of the three complexes.



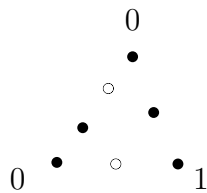
Note that the degrees of  $F^\bullet$  starts with  $c$ , then proceed in ascending order and ends with  $n - b$ .

**Lemma 2.6.** *In the degree triangle above the following holds.*

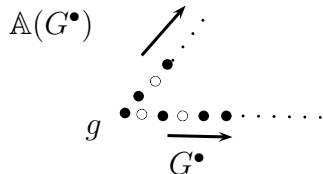
- The length, i.e. the number of circles, of the side corresponding to the set  $A$ , the degree sequence of  $F^\bullet$ , is  $a + e + 1$ . Similar relations hold for the other sides.
- The number of circles in a degree triangle is  $a + b + c + 3e$ . In particular at most a third of the circles are blank circles.
- The Koszul complexes given in Subsection 1.3 give all cases of degree triangles where there are no nondegrees, i.e. no blank circles.

*Proof.* a. This is because the number of circles is the cardinality of the interval  $[c, n - b]$  which is this number by Lemma 2.4. b. follows immediately from a. Regarding c. here are three numerical parameters for these Koszul complexes, the cardinalities of  $|A|$ ,  $|B|$  and  $|C|$ , and these correspond to  $a$ ,  $b$  and  $c$ .  $\square$

*Example 2.7.* The minimal complexes in Example 2.2 give rise to the following degree triangle.



Let  $G^\bullet$  be one of the three complexes, so  $G^\bullet$  and its Alexander dual  $\mathbb{A}(G^\bullet)$  are pure complexes. In particular they have the same initial term  $S(-n+g)^\gamma$ . We can display their degrees as



**Proposition 2.8.** *Suppose  $G^\bullet$  belongs to a triplet of pure free squarefree complexes and let  $S(-n+g)^\gamma$  be the initial term of  $G^\bullet$  and its Alexander dual  $\mathbb{A}(G^\bullet)$ . Then for each  $0 \leq v \leq n-g$ , the number of degrees of  $G^\bullet$  in the interval  $[v, n-g]$  is greater than the number of nondegrees of  $\mathbb{A}(G^\bullet)$  in the interval  $[v, n-g]$ .*

$$G^\bullet = \mathbb{D}(F^\bullet) : S(-n+c)^\alpha \rightarrow \cdots \rightarrow S(-b)^{\alpha'}$$
$$\mathbb{A}(G^\bullet) = \mathbb{A} \circ \mathbb{D}(F^\bullet) : S(-n+c)^\alpha \rightarrow \cdots S(-a)^{\alpha''}.$$

*Case 1.* In the range  $0 \leq v \leq \max\{a, b\}$  the difference  $\phi(v) - |[v, n - c]|$  is weakly decreasing. So in order to prove the statement in this range, it is enough to prove that

$$(3) \quad \phi(0) > |[0, n - c]| = n - c + 1 = a + b + e + 1.$$

$$\begin{aligned}\phi(0) &= |A| + |B| \\ &= a + e + 1 - e_A + b + e + 1 - e_B \\ &= a + b + 2 + e + e_C,\end{aligned}$$

*Case 2.* Suppose now  $v > \max\{a, b\}$ . We may as well assume that  $a \geq b$ , so  $v > a$ . Let  $c = a_0, a_1, \dots$  be the degrees of  $F^\bullet = \mathbb{D}(G^\bullet)$ , with  $S(-a_i)^{\alpha_i}$  in cohomological degree  $-i$ . The cohomology module  $H^{-i}(F^\bullet)$  is transferred to the  $-i$ 'th linear strand of  $\mathbb{A}(G^\bullet)$ . Note that if  $i > 0$ , the least nonzero degree of this cohomology module, if this module is nonzero, is  $\geq a_i + 1$ . Hence the largest degree occurring in the  $-i$ 'th linear strand of  $\mathbb{A}(G^\bullet)$ , if this strand is nonzero, is  $\leq n - a_i - 1$ .

Now if  $v - 1$  is an interior nondegree of  $\mathbb{A}(G^\bullet)$  then

$$\phi(v-1) - |[v-1, n-c]| \leq \phi(v) - |[v, n-c]|.$$

Therefore we might as well prove the statement for  $v - 1$ . Since  $a$  is a degree of  $\mathbb{A}(G^\bullet)$ , we may continue this way and in the end come to a situation where  $v - 1$  is a degree of  $\mathbb{A}(G^\bullet)$ . Let  $-l$  be its linear strand in  $\mathbb{A}(G^\bullet)$ . When  $l > 0$ , by what said above,  $v - 1 \leq n - a_l - 1$  or equivalently  $a_l \leq n - v$ . But this also holds when  $l = 0$ . Hence the degrees  $a_0, a_1, \dots, a_l$  of  $F^\bullet$  all belong to  $[c, n - v]$  and so

$$\phi(v) \geq (n - c - v + 1 - l) + (l + 1) > n - c - v + 1.$$

□

Given a natural number  $n$ . For an integer  $d$  let  $\bar{d} = n - d$ , and for a subset of integer  $D$  let  $\bar{D} = \{\bar{d} \mid d \in D\}$ .

**Definition 2.9.** A triplet of nonempty subsets  $(A, B, C)$  of  $\mathbb{N}_0$  is a *balanced degree triplet of type  $n$*  if there are integers  $0 \leq a, b, c, \leq n$  such that

1.

$$A \subseteq [c, \bar{b}], \quad B \subseteq [a, \bar{c}], \quad C \subseteq [b, \bar{a}]$$

and the endpoints of each interval are in the respective subsets  $A, B$  or  $C$ .

2. Let  $e_A$  be the cardinality of  $[c, \bar{b}] \setminus A$  and correspondingly define  $e_B$  and  $e_C$ .

Then  $n = a + b + c + e_A + e_B + e_C$ .

3.  $A$  and  $\bar{B}$  are balanced with respect to the common endpoint  $c$ , i.e. for each  $c \leq v \leq n$ , the number of elements of  $[c, v]$  in  $A$  is greater than the number of elements of  $[c, v]$  not in  $\bar{B}$ . Similarly for  $B$  and  $\bar{C}$  with respect to  $a$  and  $C$  and  $\bar{A}$  with respect to  $b$ .

**Conjecture 2.10.** *a. For each balanced degree triplet  $(A, B, C)$  of type  $n$ , there exists a triplet of pure free squarefree complexes over the polynomial ring in  $n$  variables whose degree sequences are given by  $A, B$ , and  $C$ .*

*b. The Betti numbers of this triplet of complexes are uniquely determined by the degree triplet, up to common scalar multiple.*

### 3. CONSTRAINTS ON THE BETTI NUMBERS

In this section we give linear equations fulfilled by the Betti numbers in a triplet of pure complexes. The number of equations is one less than the number of Betti numbers, so we expect a unique set of Betti numbers up to multiplication by a common scalar. We prove that this is the case, provided part a. of Conjecture 2.10 holds. In other words we prove that part a. of the conjecture implies part b.

**3.1. Some elementary relations for binomial coefficients.** For nonnegative integers  $p$  we have the binomial coefficient  $\binom{x}{p}$ . When  $p$  is a negative integer we set this coefficient to be zero. The following identities hold in  $\mathbb{Q}[x, y]$  and are repeatedly used in the proof of the below lemma.

1.  $\binom{x+y}{p} = \sum_{i=0}^p \binom{x}{p-i} \binom{y}{i}$ , [13, Ex. 4.3.3].
2.  $\binom{x}{p} = (-1)^p \binom{p-1-x}{p}$ .

Let  $A = (a_{ij})$  be the  $(n+1) \times (n+1)$ -matrix with  $a_{ij} = (-1)^j \binom{n-j}{i}$  for  $i, j = 0, \dots, n$ . For instance when  $n = 2$  this is the matrix

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

**Lemma 3.1.**  $A^3 = (-1)^n \cdot I$

*Proof.* First we show that  $A^2 = (b_{ij})$  where  $b_{ij} = (-1)^j \binom{j}{n-i}$ . For instance when  $n = 2$  this is

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}.$$

The  $i$ 'th row in  $A$  is

$$\binom{n}{i}, -\binom{n-1}{i}, \binom{n-2}{i}, \dots$$

Now

$$\binom{n-j}{i} = \binom{n-j}{n-j-i} = (-1)^{n-i-j} \binom{-i-1}{n-j-i}.$$

The  $i$ 'th row of  $A$  is then  $(-1)^{n-i}$  multiplied with:

$$\binom{-i-1}{n-i}, \binom{-i-1}{n-1-i}, \binom{-i-1}{n-2-i}, \dots$$

The  $j$ 'th column in  $A$  is  $(-1)^j$  multiplied with the following:

$$\binom{n-j}{0}, \binom{n-j}{1}, \binom{n-j}{2}, \dots$$

From this  $b_{ij}$  is  $(-1)^{n-i-j}$  multiplied with

$$\begin{aligned} & \binom{n-j}{0} \binom{-i-1}{n-i} + \binom{n-j}{1} \binom{-i-1}{n-1-i} + \dots \\ &= \binom{n-j-i-1}{n-i} = (-1)^{n-i} \binom{j}{n-i}. \end{aligned}$$

Hence  $b_{ij} = (-1)^j \binom{j}{n-i}$ .

To find  $A^3$  note that the  $i$ 'th row in  $A^2$  is

$$\binom{0}{n-i}, -\binom{1}{n-i}, \binom{2}{n-i}, \dots$$

Note that

$$\binom{j}{n-i} = \binom{j}{j+i-n} = (-1)^{j+i-n} \binom{i-n-1}{j+i-n}.$$

Hence row  $i$  is  $(-1)^{n-i}$  multiplied with

$$\binom{i-n-1}{i-n}, \binom{i-n-1}{i-n+1}, \binom{i-n-1}{i-n+2}, \dots$$

The  $j$ 'th column in  $A$  is  $(-1)^j$  multiplied with

$$\binom{n-j}{0}, \binom{n-j}{1}, \binom{n-j}{2}, \dots$$

Since

$$\binom{n-j}{i} = \binom{n-j}{n-j-i}$$

this column becomes

$$\binom{n-j}{n-j}, \binom{n-j}{n-j-1}, \binom{n-j}{n-j-2}, \dots$$

The first nonzero position in the  $i$ 'th row is  $n - i$ . The last nonzero position in the  $j$ 'th column is  $n - j$ . Hence if  $n - j < n - i$ , equivalently  $i < j$ , the product of the  $i$ 'th row and  $j$ 'th column is zero. On the other hand if  $i \geq j$  the product is  $(-1)^{n-i-j}$  multiplied with

$$\begin{pmatrix} i-1-j \\ i-j \end{pmatrix} = (-1)^{i-j} \begin{pmatrix} 0 \\ i-j \end{pmatrix} = \begin{cases} 1 & i=j \\ 0 & i>j \end{cases}.$$

Hence we obtain  $A^3 = (-1)^n \cdot I$ .  $\square$

**3.2. Linear equations for the Betti numbers.** Let  $F^\bullet$  be the pure free squarefree complex

$$(4) \quad F^\bullet : S(-a_0)^{\alpha_0} \leftarrow S(-a_1)^{\alpha_1} \leftarrow \cdots \leftarrow S(-a_r)^{\alpha_r}.$$

Let  $\hat{\alpha}_{a_i} = (-1)^{l(a_i)} \cdot \alpha_i$  where  $l(a_i)$  is the linear strand containing the term  $S(-a_i)^{\alpha_i}$ , be the *signadjusted* Betti numbers. We set  $\hat{\alpha}_d = 0$  if  $d$  is not a degree of  $F^\bullet$ . Note that these signadjusted Betti numbers are parametrized by the internal degrees. Note also that

$$(-1)^{l(a_i)} = (-1)^{i+a_i+l(a_0)-a_0}.$$

Assume also that  $\mathbb{A} \circ \mathbb{D}(F^\bullet)$  is a pure complex

$$S(-b_0)^{\beta_0} \leftarrow \cdots \leftarrow S(-b_{r'})^{\beta_{r'}}.$$

Recall that the  $i$ 'th homology module of  $F^\bullet$  is transferred to the  $i$ 'th linear strand of  $\mathbb{A} \circ \mathbb{D}(F^\bullet)$ .

Suppose the  $i$ 'th homology module of  $F^\bullet$  is nonzero and let  $d$  be a degree for which the  $d$ 'th graded part of this module is nonzero. This module is squarefree and the dimension of its squarefree part in degree  $d$  (recall this notion in Subsection 1.1) is

$$(5) \quad (-1)^{i+l(a_0)-a_0} [\alpha_{a_0} \binom{n-a_0}{n-d} - \alpha_{a_1} \binom{n-a_1}{n-d} + \alpha_{a_2} \binom{n-a_2}{n-d} + \cdots].$$

This will be equal to  $(-1)^i \hat{\beta}_{n-d}$ . The equations (5) when  $d$  varies are then the same as

$$(6) \quad \hat{\beta} = A \cdot \hat{\alpha}.$$

If furthermore  $(\mathbb{A} \circ \mathbb{D})^2(F^\bullet)$  is pure we get in the same way

$$(7) \quad \hat{\gamma} = A \cdot \hat{\beta}$$

$$(8) \quad \hat{\hat{\alpha}} = A \cdot \hat{\gamma},$$

where  $\hat{\hat{\alpha}} = (-1)^n \hat{\alpha}$  due to the shift  $n$  of linear strands of the functor  $(\mathbb{A} \circ \mathbb{D})^3$ .

If  $F^\bullet$ ,  $\mathbb{A} \circ \mathbb{D}(F^\bullet)$  and  $(\mathbb{A} \circ \mathbb{D})^2(F^\bullet)$  are all pure then the clearly the following equations hold

$$(9) \quad \hat{\alpha}_i = 0 \text{ for all nondegrees } i \text{ of } F^\bullet \text{ in } [0, n],$$

$$(10) \quad \hat{\beta}_i = 0 \text{ for all nondegrees } i \text{ of } (\mathbb{A} \circ \mathbb{D})(F^\bullet) \text{ in } [0, n],$$

$$(11) \quad \hat{\gamma}_i = 0 \text{ for all nondegrees } i \text{ of } (\mathbb{A} \circ \mathbb{D})^2(F^\bullet) \text{ in } [0, n].$$

In addition we must have the equations (6), (7) and (8) above (where any two of these determine the third by Lemma 3.1).



**Lemma 3.2.** *The equations  $\hat{\alpha}_i = 0$  for  $i = 0, \dots, c-1$  are equivalent to the equations  $\hat{\beta}_{n-i} = 0$  for  $i = 0, \dots, c-1$ .*

*Similarly the equations  $\hat{\beta}_i = 0$  and  $\hat{\gamma}_{n-i} = 0$  for  $i = 0, \dots, a-1$  are equivalent, and  $\hat{\gamma}_i = 0$  and  $\hat{\alpha}_{n-i} = 0$  for  $i = 0, \dots, b-1$  are equivalent.*

*Proof.* This is due to the transition matrix  $A$  having the triangular form

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ * & * & * & 0 & \cdots \\ * & * & 0 & \cdots & \\ * & 0 & \cdots & & \end{bmatrix}.$$

□

**Corollary 3.3.** *Given a balanced degree triangle. The  $3n + 3$  signadjusted Betti numbers  $\hat{\alpha}_i, \hat{\beta}_i, \hat{\gamma}_i$ ,  $i = 0, \dots, n$  fulfil the equations (6), (7), (8), (9), (10), and (11), which may be reduced to  $3n + 2$  natural equations.*

*Remark 3.4.* We expect these equations to be linearly independent. Hence there would be a unique solution up to scalar multiple.

*Proof.* There are  $c+b+e_A$  equations of the form  $\hat{\alpha}_i = 0$ . Similarly there are  $a+c+e_B$  equations of the form  $\hat{\beta}_i = 0$ , and  $a+b+e_C$  equations of the form  $\hat{\gamma}_i = 0$ . This give a total of  $2a+2b+2c+e$  equations. However by the above Lemma 3.2 there are  $a+b+c$  dependencies among them, giving  $a+b+c+e = n$  equations. In addition the transition equations (7) and (8) give  $2n+2$  further equations, a total of  $3n+2$ . □

The complex  $F^\bullet$  is

$$S(-n+b)^{\alpha_{r_0}} \rightarrow \cdots \rightarrow S(-c)^{\alpha_0}.$$

Its Alexander dual  $\mathbb{A}(F^\bullet)$  equals (up to translation)  $\mathbb{D} \circ (\mathbb{A} \circ \mathbb{D})^2 F^\bullet$  which is

$$S(-n+b)^{\gamma_0} \rightarrow \cdots \rightarrow S(-a)^{\gamma_{r_2}}.$$

(Note that  $\gamma_0 = \alpha_{r_0}$ .) Let  $v_1 < \cdots < v_{e_C}$  be the internal nondegrees of  $\mathbb{A}(F^\bullet)$ .

The complex  $\mathbb{D}(F^\bullet)$  is

$$S(-n+c)^{\alpha_0} \rightarrow \cdots \rightarrow S(-b)^{\alpha_{r_0}}$$

and then its Alexander dual  $\mathbb{A} \circ \mathbb{D}(F^\bullet)$  is

$$S(-n+c)^{\beta_{r_1}} \rightarrow \cdots \rightarrow S(-a)^{\beta_0}.$$

(Note that  $\beta_{r_1} = \alpha_0$ .) Let  $u_1 < \cdots < u_{e_B}$  be the internal nondegrees of  $\mathbb{A} \circ \mathbb{D}(F^\bullet)$ .

**Proposition 3.5.** *Given a triplet of pure free squarefree complexes. Let  $a_0 < \cdots < a_r$  be the degrees of the first complex  $F^\bullet$  and  $\bar{a}_r < \cdots < \bar{a}_0$  be the degrees of the dual  $\mathbb{D}(F^\bullet)$ . By the transition equations (6), (7), and (8), the equations (9), (10), and (11) are equivalent to the following equations for the  $(r+1)$  nonzero Betti numbers*

$\alpha_i$ .

$$(12) \quad \alpha_0 \binom{a_0}{v_i} - \alpha_1 \binom{a_1}{v_i} + \cdots + (-1)^r \alpha_r \binom{a_r}{v_i} = 0, \quad i = 1, \dots, e_C$$

$$(13) \quad \alpha_0 \binom{\bar{a}_0}{u_i} - \alpha_1 \binom{\bar{a}_1}{u_i} + \cdots + (-1)^r \alpha_r \binom{\bar{a}_r}{u_i} = 0, \quad i = 1, \dots, e_B$$

$$(14) \quad \alpha_0 \binom{a_0}{j} - \alpha_1 \binom{a_1}{j} + \cdots + (-1)^r \alpha_r \binom{a_r}{j} = 0, \quad j = 0, \dots, a-1$$

The total number of these equations  $e_C + e_B + a$ , equals  $r$ .

*Proof.* The last part is because

$$r + e_A = n - b - c, \text{ and } a + b + c + e_A + e_B + e_C = n.$$

By the transition equation (6) the set of equations (13) is equivalent to  $\hat{\beta}_{u_i} = 0$  for each nondegree  $u_i$  of  $\mathbb{A} \circ \mathbb{D}(F^\bullet)$  in the interval  $[a, n - c]$ . The vanishing of  $\hat{\beta}_j$  for  $j \in [n - c + 1, n]$  is by Lemma 3.2 equivalent to  $\hat{\alpha}_j = 0$  for  $j \leq c - 1$ . The vanishing of  $\hat{\beta}_j$  for  $j \in [0, a - 1]$  is again by the transition equation (6) equivalent to the equations (14).

In the same way the vanishing of  $\hat{\gamma}_j$  for each nondegree  $j$  of  $(\mathbb{A} \circ \mathbb{D})^2(F^\bullet)$  in the interval  $[b, n - a]$  is equivalent to the equations (12). The vanishing of  $\hat{\gamma}_j$  for  $j$  in  $[n - a + 1, n]$  are by Lemma 3.2 equivalent to the vanishing of  $\hat{\beta}_j$  for  $j$  in  $[0, a - 1]$  which are again equivalent to equations (14).  $\square$

We also get corresponding equations for the  $\beta_i$  and the  $\gamma_i$ .

**Corollary 3.6.** *Given a balanced degree triplet. If the equations (12), (13), and (14) for the Betti numbers  $\alpha_i$  of the pure complex  $F^\bullet$  has a  $k$ -dimensional solution set, then the corresponding equations for the Betti numbers  $\beta_i$  of the pure complex  $\mathbb{A} \circ \mathbb{D}(F^\bullet)$  has a  $k$ -dimensional solution set, and similarly for the Betti numbers  $\gamma_i$  of  $(\mathbb{A} \circ \mathbb{D})^2(F^\bullet)$ .*

*Proof.* By the transition equations (6), (7), and (8), all these equation systems are equivalent to the equations (9), (10), and (11).  $\square$

**3.3. Uniqueness of Betti numbers.** Given a balanced degree triplet  $\Delta = (A, B, C)$ . The set  $A$  is a subset of  $[c, n - b]$ , containing the end points of this interval. Let us suppose that there is an internal nondegree of  $A$ , i.e.  $A$  is a proper subset of  $[c, n - b]$ . Let  $A$  contain  $[c, c + t - 1]$  but not  $c + t$ . The set  $B$  is a subset of  $[a, n - c]$  containing the endpoints. Let  $s \geq 1$  be maximal such that  $\bar{B} \subseteq [c, n - b]$  is disjoint from the interval  $[c + 1, c + s - 1]$ . Since the degree triangle is balanced we have  $s \leq t$ . Let  $\Delta' = (A', B', C)$  where

$$A' = A \cup \{c + t\} \setminus [c, c + s - 1], \quad B' = B \setminus \{c\}.$$

**Lemma 3.7.** *If  $\Delta$  is a balanced degree triplet, then  $\Delta'$  is a balanced degree triplet.*

*Proof.* If  $\Delta$  has  $e$  internal nondegrees, then clearly  $\Delta'$  has  $e - s$  internal nondegrees. (We remove one nondegree from  $A$  and  $s - 1$  from  $C$ .) Since  $\Delta'$  has parameters  $c + s, b$  and  $a$ , the equation  $a + b + c + e = n$  continues to hold when passing from  $\Delta$  to  $\Delta'$ . Also viewing  $\Delta'$  from each of its three corners it is clearly also balanced,  $\square$

**Proposition 3.8.** *Let  $\Delta$  be a balanced triplet, with an internal nondegree on one of its sides, say side  $A$ . Then if i) there exists a triplet of pure free squarefree complexes for the degree triplet  $\Delta'$ , and ii) the equation system for  $\Delta'$  has a one-dimensional solution, then the equation system for  $\Delta$  also has a one-dimensional solution.*

*Proof.* Let  $X$  be the coefficient matrix for the system of equations given in Proposition 3.5 for the Betti numbers  $\alpha_i$ , of a pure complex associated to  $A$  in the degree triplet  $\Delta$ . Let  $X'$  be the corresponding coefficient matrix for the triplet  $\Delta'$ . By hypothesis the solution set of  $X'$  is one-dimensional. The coordinates of a solution vector  $(\alpha'_0, \dots, \alpha'_{r'})$  may be taken as the minors of the matrix  $X'$ . By hypothesis there exists a pure free squarefree complex  $F'^{\bullet}$ , part of a triplet, whose Betti numbers are a multiple of this solution vector, and hence all the  $\alpha'_i$  will be nonzero.

Now note that the columns in  $X'$  has columns parametrizes by the degrees of  $A'$  in  $[c + s, n - b]$ . These are exactly the degrees of  $A$  which are in  $[c + s, n - b]$ , together with the degree  $c + t$ . Write the coefficient matrix  $X$  such that equations (13) are the first rows and the equations (12) the second group of rows, and the (14). If we remove the first column in  $X$ , indexed by  $c$ , then  $X$  will have a form

$$\begin{bmatrix} T & 0 \\ Z & Y \end{bmatrix}$$

where  $T$  is a triangular matrix of size  $(s - 1) \times (s - 1)$ . If we remove the column of  $X'$  indexed by  $c + t$ , we will simply get the matrix  $Y$ . Hence the determinant of  $Y$ , which is one of the  $\alpha'_i$ , is nonzero. So the matrix  $X$  will have full rank, and hence a one-dimensional solution set.  $\square$

We then get the following.

**Theorem 3.9.** *Part a. of Conjecture 2.10 implies part b. of the conjecture: The Betti numbers of triplets of pure complexes of free squarefree modules, associated to a balanced degree triplet, are uniquely determined up to common scalar multiple.*

*Proof.* This follows from the previous proposition once we know it is true for the induction start. And the induction start is a degree triplet with no nondegrees. But in any degree triplet where all nondegrees are on only one edge, the uniqueness of Betti numbers follows by the Herzog-Kühl equations, see [11, Sec. 1.3], since if this edge corresponds to a complex  $F^{\bullet}$ , then this complex is a resolution of a Cohen-Macaulay module, Lemma 1.9. The uniqueness of all Betti numbers up to common scalar multiple follows by the transition equations (6), (7), and (8).  $\square$

#### 4. CONSTRUCTION OF TRIPLETS WHEN THE INTERNAL NONDEGREES ARE ON ONLY ONE SIDE OF THE DEGREE TRIANGLE

In this section we construct triplets of pure squarefree complexes in the case that two of the complexes are linear. These correspond to degree triangles where two of the sides only consists of degrees (filled circles).

**4.1. Auxiliary results on subspaces of vector spaces.** Let  $E$  be a vector space and  $E_1, \dots, E_r$  subspaces of  $E$ . For  $I$  a subset of  $[r] = \{1, \dots, r\}$  we let  $E_I$  be the intersection  $\cap_{i \in I} E_i$ .

**Lemma 4.1.** *Suppose  $E_1, \dots, E_r$  are general subspaces of  $E$  of codimension one, where  $r \leq \dim_{\mathbb{k}} E$ . Then the  $E_{[r] \setminus \{i\}}$  as  $i$  varies through  $i = 1, \dots, r$ , generate  $E$ .*

*Proof.* By dividing out by  $E_{[r]}$  we may as well assume that  $r = \dim_{\mathbb{K}} E$ . Then each  $E_{[r] \setminus \{i\}}$  corresponds to a one-dimensional vector space. To construct the  $E_i$  we may choose general vectors  $v_1, \dots, v_r$  and let  $E_i$  be spanned by the  $(r-1)$ -subsets of this  $r$ -set we get by successively omitting the  $v_i$ .  $\square$

**Lemma 4.2.** *Let  $E_i$  be a subspace of  $E$  of codimension  $e_i$  for  $i = 1, \dots, r$ . Suppose for each proper subsets  $J$  of  $I$  that the codimension of  $E_J$  is  $\sum_{i \in J} e_i$ . If  $\text{codim } E_I < \sum_{i \in I} e_i$ , then the  $E_{I \setminus \{i\}}$  do not generate  $E$  as  $i$  varies through  $I$ .*

*Proof.* Let the codimension of  $E_I$  be  $(\sum_{i \in I} e_i) - r$  where  $r > 0$ . By dividing out by  $E_I$  we may assume  $E_I = 0$  and so this number is the dimension of  $E$ . Then the dimension of  $E_{I \setminus \{i\}}$  is  $e_i - r$ , and so if  $|I| \geq 2$  these cannot generate the whole space  $E$ .  $\square$

**Notation.** We shall in the following denote by  $S^r(E)$  the  $r$ 'th symmetric power of  $E$  and by  $D^r(E)$  the  $r$ 'th divided power of  $E$ . Also let  $\tilde{D}^r(E) = \wedge^{\dim_{\mathbb{K}} E} E \otimes_{\mathbb{K}} D^r(E)$ .

**4.2. Construction of tensor complexes.** We start with a degree triplet  $(A, B, C)$  where  $B = [a, \bar{c}]$  and  $C = [b, \bar{a}]$  are intervals, i.e. contain no nondegrees. We partition the complement of  $A$  in  $[0, n]$  into successive intervals

$$[u_0+1, u_0+w_0-1], [u_1+1, u_1+w_1-1], \dots, [u_r+1, u_r+w_r-1], [u_{r+1}+1, u_{r+1}+w_{r+1}-1],$$

where for the first and last interval we have  $u_0 = -1, w_0 = c+1$  and  $u_{r+1} = n-b$  and  $w_{r+1} = b+1$ , and for the middle intervals  $c \leq u_1, u_i + w_i \leq u_{i+1}$ , and  $u_r + w_r \leq n-b$ . Let  $W_i$  be a vector space of dimension  $w_i$ , and  $W = \otimes_{i=0}^{r+1} W_i$ . Denote by  $\vec{W}$  the tuple  $(W_0, \dots, W_{r+1})$ . Let  $V$  be a vector space of dimension  $n$  and  $S(V \otimes W^*)$  the symmetric algebra. In the language of [1, Sec. 5],  $(0; u_0, \dots, u_{r+1})$  is a *pinching weight* for  $V, \vec{W}$ .

Berkesch et.al. [1], construct a resolution  $F^\bullet(V; \vec{W})$  of pure free  $S(V \otimes W^*)$ -modules with degree sequence  $A$  such that the term with free generators of degree  $d \in A$  has the form:

$$(15) \quad \bigwedge^d V \otimes (\otimes_{d \leq u_i} S^{u_i-d}(W_i)) \otimes (\otimes_{d \geq u_i+w_i} \tilde{D}^{d-u_i+w_i}(W_i)) \otimes S(V \otimes W^*).$$

This complex is a resolution of a Cohen-Macaulay module and is equivariant for the group

$$GL(V) \times GL(W_0) \times \dots \times GL(W_{r+1}).$$

The construction of this complex follows the method of Lascoux, presented in [17, Sec. 5.1]. Let  $\mathbb{P}(\vec{W})$  be the product  $\mathbb{P}(W_0) \times \dots \times \mathbb{P}(W_{r+1})$ . There is a tautological sequence:

$$0 \rightarrow \mathcal{S} \rightarrow W \otimes \mathcal{O}_{\mathbb{P}(\vec{W})} \rightarrow \mathcal{O}_{\mathbb{P}(\vec{W})}(1, \dots, 1) \rightarrow 0.$$

Dualizing this sequence and tensoring with  $V$  we get a sequence (let  $\mathcal{Q} = \mathcal{S}^*$ )

$$0 \rightarrow V \otimes \mathcal{O}_{\mathbb{P}(\vec{W})}(-1, \dots, -1) \rightarrow V \otimes W^* \otimes \mathcal{O}_{\mathbb{P}(\vec{W})} \rightarrow V \otimes \mathcal{Q} \rightarrow 0.$$

Constructing the affine bundles over  $\mathbb{P}(\vec{W})$  of the last two terms in this complex, we get a diagram

$$\begin{array}{ccc} Z = \mathbb{V}(V \otimes \mathcal{Q}) & \hookrightarrow & \mathbb{V}((V \otimes W^*) \otimes \mathcal{O}_{\mathbb{P}(\vec{W})}) = \mathbb{V}(V \otimes W^*) \times \mathbb{P}(\vec{W}) \\ \downarrow & & \downarrow \pi \\ Y & \hookrightarrow & \mathbb{V}(V \otimes W^*) \end{array}$$

where  $Y$  is the image of  $Z$  by the projection  $\pi$ . The projection of the structure sheaf  $\pi_*(\mathcal{O}_Z)$  is the sheaf on the affine space  $Y$  associated to the  $S(V \otimes W^*)$ -module  $H^0(\mathbb{P}(\vec{W}), \text{Sym}(V \otimes \mathcal{Q}))$ .

Let  $p$  be the projection of  $\mathbb{V}(V \otimes W^*) \times \mathbb{P}(\vec{W})$  to the second factor. Let  $\mathcal{L}$  be the line bundle  $\mathcal{O}_{\mathbb{P}(\vec{W})}(u_0, \dots, u_{r+1})$  on  $\mathbb{P}(\vec{W})$ . Then  $M = H^0(\mathbb{P}(\vec{W}), \text{Sym}(V \otimes \mathcal{Q}) \otimes p^*\mathcal{L})$  is an  $S(V \otimes W^*)$ -module and the complex  $F(V; \vec{W})$  is a resolution of this module. The sheafification of this module on the affine space is in fact  $\pi_*(\mathcal{O}_Z \otimes p^*\mathcal{L})$ .

**Fact.**  $\dim Y = \dim Z$ . This is argued for in [1], see for instance the proof of Proposition 3.3. First note that

$$\dim Z = \dim \mathbb{P}(\vec{W}) + n \cdot \text{rk} \mathcal{Q}.$$

Since  $F(V; \vec{W})$  is a resolution of a module supported on  $Y$ , the length of this resolution is at least the codimension of  $Y$ . Hence

$$\begin{aligned} \dim Y &\geq n \dim_{\mathbb{k}} W - |A| + 1 \\ &= n \dim_{\mathbb{k}} W - n + \sum_i (w_i - 1). \end{aligned}$$

Since  $\text{rk} \mathcal{Q} = \dim_{\mathbb{k}} W - 1$  and  $\dim \mathbb{P}(\vec{W}) = \sum_i (w_i - 1)$  we get  $\dim Y \geq \dim Z$  and we obviously also have the opposite inequality.

**4.3. Degeneracy loci of bundles.** Let  $\mathcal{E}$  be a vector bundle, i.e. a locally free sheaf of finite rank  $e$ , on a scheme  $S$ . Let  $T$  be a subspace of the sections  $H^0(S, \mathcal{E})$ . The map  $T \otimes_{\mathbb{k}} \mathcal{O}_S \rightarrow \mathcal{E}$  defines a map and an exact sequence

$$T \otimes_{\mathbb{k}} \text{Sym}(\mathcal{E}) \rightarrow \text{Sym}(\mathcal{E}) \rightarrow \mathcal{R} \rightarrow 0$$

where the cokernel  $\mathcal{R}$  is a quasi-coherent sheaf of  $\mathcal{O}_S$ -algebras. The space  $T$  gives global sections of the affine bundle  $\mathbb{V} = \mathbb{V}_S(\mathcal{E})$  and they generate a sheaf of ideals of  $\mathcal{O}_{\mathbb{V}}$  defining a subscheme  $\mathcal{X} = \text{Spec}_{\mathcal{O}_S} \mathcal{R}$ .

Now we may stratify  $S$  according to the rank of the map  $T \otimes_{\mathbb{k}} \mathcal{O}_S \rightarrow \mathcal{E}$ . Let  $U_i$  be the open subset where the rank is  $\geq \dim_{\mathbb{k}} T - i$ . Then if  $x \in U_i \setminus U_{i-1}$  we get an exact sequence

$$T \otimes \text{Sym}(\mathcal{E}_{k(x)}) \rightarrow \text{Sym}(\mathcal{E}_{k(x)}) \rightarrow \mathcal{R}_{k(x)} \rightarrow 0$$

where  $\mathcal{R}_{k(x)}$  is the quotient symmetric algebra generated by a vector space of dimension  $e - t + i$ . Hence the fiber  $\mathcal{X}_{k(x)}$  has dimension  $e - t + i$ . We observe that the dimension of  $\mathcal{X}$  is less than or equal to the maximum of

$$(16) \quad \max\{\dim(S \setminus U_{i-1}) + e - t + i\}.$$

We adapt this to the situation of Subsection 4.2 so  $S = \mathbb{P}(\vec{W})$ . Let  $V$  be a vector space with a basis  $x_1, \dots, x_n$  and  $\mathcal{E} = V \otimes_{\mathbb{k}} \mathcal{Q}$ . For each  $x_i$  chose a general subspace  $E_i \subseteq W^*$  of codimension one. Let  $T$  be the space of sections

$$\bigoplus_i x_i \otimes E_i \subseteq \bigoplus_i x_i \otimes W^* = V \otimes W^*.$$

Note that the dimension of  $T$  equals the rank of  $V \otimes_{\mathbb{k}} \mathcal{Q}$ .

**Proposition 4.3.** *The locus where the composition*

$$\alpha : T \otimes_{\mathbb{k}} \mathcal{O}_{\mathbb{P}(\vec{W})} \hookrightarrow V \otimes_{\mathbb{k}} W^* \otimes_{\mathbb{k}} \mathcal{O}_{\mathbb{P}(\vec{W})} \rightarrow V \otimes \mathcal{Q}$$

*degenerates to rank  $\dim_{\mathbb{k}} T - i$ , has codimension  $\geq i$ .*

*Proof.* The map  $\alpha$  is the direct sum of maps

$$\alpha_i : E_i \otimes_{\mathbb{k}} \mathcal{O}_{\mathbb{P}(\vec{W})} \rightarrow \mathcal{Q}.$$

The rank of  $\alpha$  is then the sum of the ranks of these maps. Let  $K$  be a subset of  $[n]$ , and  $E_K = \bigcap_{i \in K} E_i$ . Since  $\mathcal{Q}$  is generated by its global sections, and  $\dim_{\mathbb{k}} E_K = \text{rk} \mathcal{Q} - |K| + 1$ , the locus where the image of  $E_K \otimes_{\mathbb{k}} \mathcal{O}_{\mathbb{P}(\vec{W})} \xrightarrow{\alpha_K} \mathcal{Q}$  has codimension

$$(17) \quad \geq c + \text{rk} \mathcal{Q} - \dim_{\mathbb{k}} E_K = c + |K| - 1$$

has codimension greater or equal than

$$(18) \quad \begin{aligned} c(c + \text{rk} \mathcal{Q} - \dim_{\mathbb{k}} E_K) &= c(c + |K| - 1) \\ &\geq c + |K| - 1 \end{aligned}$$

when  $c \geq 1$ . This last condition is by (17) equivalent to the image of  $\alpha_K$  being of codimension  $\geq |K|$ .

Let  $X' \subseteq \mathbb{P}(\vec{W})$  be the locus of points where  $E_K \otimes_{\mathbb{k}} \mathcal{Q}_{\mathbb{P}(\vec{W})} \xrightarrow{\alpha_K} \mathcal{Q}$  degenerates to codimension  $\geq \sum_{i \in K} q_i$  where each  $q_i \geq 1$ . Since  $\sum_{i \in K} q_i = c + |K| - 1 \geq |K|$  we have  $c \geq 1$  and so by (18) we have  $\text{codim } X' \geq \sum_{i \in K} q_i$ .

Now let  $X$  be the locus of points where the image of  $\alpha_{i, \mathbb{k}(x)}$  has codimension  $\geq q_i$  for  $i \in K$ , where each  $q_i \geq 1$ . We want to show that  $X \subseteq X'$ . This will show that  $\text{codim } X \geq \sum_{i \in K} q_i$  which is what we want to prove.

Let  $x \in \mathbb{P}(\vec{W})$  be a point where the image of  $\alpha_{i, \mathbb{k}(x)}$  has codimension  $q_i$ . Clearly for any  $I \subseteq [n]$ , the image of  $\alpha_{I, \mathbb{k}(x)}$  is contained in  $\bigcap_{i \in I} \text{im } \alpha_{i, \mathbb{k}(x)}$ . Suppose there is  $I \subseteq K$  such that this intersection in  $\mathcal{Q}_{\mathbb{k}(x)}$  does not have codimension  $\sum_{i \in I} q_i$  and let  $I$  be minimal such in  $K$ . But the image of  $E_{I \setminus \{j\}}$  is contained in the intersection  $\bigcap_{i \in I \setminus \{j\}} \text{im } \alpha_{i, \mathbb{k}(x)}$  and these do not generate  $\mathcal{Q}_{\mathbb{k}(x)}$  by Lemma 4.2. If  $|I| \leq \dim_{\mathbb{k}} W = \text{rk} \mathcal{Q} + 1$  this is not possible since the  $E_{I \setminus \{j\}}$  generate  $W^*$  by Lemma 4.1, and the map  $W^* \otimes \mathcal{O}_{\mathbb{P}(\vec{W})} \rightarrow \mathcal{Q}$  is surjective. Hence in this case the image of  $E_K \xrightarrow{\alpha_{K, \mathbb{k}(x)}} \mathcal{Q}_{\mathbb{k}(x)}$  must be of codimension  $\geq \sum_{i \in I} q_i$ . That  $|I| \geq \text{rk} \mathcal{Q} + 2$  is not possible by minimality of  $I$  since then  $\bigcap_{i \in I \setminus \{j\}} \text{im } \alpha_{i, \mathbb{k}(x)}$  cannot have codimension  $\geq \sum_{i \in I \setminus \{j\}} q_i \geq \text{rk} \mathcal{Q} + 1$  (actually for any  $j$ ). This proves the proposition.  $\square$

**Corollary 4.4.** *The dimension of  $\mathcal{X} = \text{Spec}_{\mathcal{O}_{\mathbb{P}(\vec{W})}} \mathcal{R}$ , the subscheme of  $\mathbb{V}_{\mathbb{P}(\vec{W})}(\mathcal{E})$  defined by the vanishing of the sections  $T$ , is less than or equal to the dimension of  $\mathbb{P}(\vec{W})$ .*

*Proof.* This follows by the above Proposition 4.3 and the expression for the dimension given by (16).  $\square$

**4.4. Construction of pure free squarefree resolutions from tensor complexes.** Recall that  $x_1, \dots, x_n$  is a basis for  $V$ . Consider the map

$$(19) \quad V \otimes W^* = \bigoplus_{i=1}^n x_i \otimes W^* \rightarrow \bigoplus_{i=1}^n x_i \otimes (W^*/T_i) \cong V.$$

It induces a homomorphism of algebras

$$S(V \otimes W^*) \rightarrow S(V).$$

Recall the pure resolution  $F^\bullet(V; \vec{W})$  of Subsection 4.2 whose degree sequence is given by the set  $A$ .

**Proposition 4.5.** *The complex*

$$F^\bullet = F^\bullet(V; \vec{W}) \otimes_{S(V \otimes W^*)} S(V)$$

*is a pure free squarefree resolution of a Cohen-Macaulay squarefree  $S(V)$ -module. Its degree sequence is  $A$ .*

*Remark 4.6.* The essential thing about the tensor complex  $F(V; \vec{W})$  that makes this construction work is that in the generators of its free modules in (15), the only representations of  $V$  that occur are the exterior forms  $\wedge^d V$ . Choosing a basis  $x_1, \dots, x_n$  for  $V$ , this is generated by (squarefree) exterior monomials. This is why the tensor complexes are “tailor made” for our construction.

*Remark 4.7.* In our construction we could equally well have used the  $GL(F) \times GL(G)$ -equivariant complex of [7, Sec.4]. Again in the generators of the free modules, the  $\wedge^d F$  are the only representations of  $F$  that occur. In contrast all kinds of irreducible representations of  $G$  are involved, and it also only works in char.  $\mathbb{k} = 0$ , which is why we focus on the tensor complexes of [1].

From this we obtain as a corollary the following.

**Theorem 4.8.** *For a balanced degree triplet  $(A, B, C)$  of type  $n$ , where  $B$  and  $C$  are intervals, i.e. the only internal nondegrees are in the interval associated to  $A$ , there exists a triplet of pure free squarefree modules over the polynomial ring in  $n$  variables whose degree sequences are given by  $A, B$  and  $C$ .*

*Proof of Theorem 4.8.* Let the endpoints of  $A$  be  $c$  and  $n - b$ . All nondegrees of this triplet are in  $[c, n - b]$ , so the number  $e$  of such is the cardinality of  $[c, n - b] \setminus A$ . Since  $a + b + c + e = n$  we see that  $a$  is determined. Hence  $B$  and  $C$  are determined by  $A$ . By Lemma 1.9  $(\mathbb{A} \circ \mathbb{D})(F^\bullet)$  and  $(\mathbb{A} \circ \mathbb{D})^2(F^\bullet)$  are both linear and so have degrees given by  $B$  and  $C$ .  $\square$

*Remark 4.9.* In the forthcoming paper [12] we consider the construction of triplets of pure complexes in general. We transfer Conjecture 2.10 to a conjecture on the existence of certain complexes of coherent sheaves on projective spaces. In the case of the above theorem, these complexes reduce to a single coherent sheaf, the line bundle  $\mathcal{O}_{\mathbb{P}(\vec{W})}(u_0, \dots, u_{r+1})$  on the Segre embedding of  $\mathbb{P}(\vec{W})$  in the projective space  $\mathbb{P}(W)$ .

*Proof of Proposition 4.5.* Let  $Z'$  be the pullback in the diagram

$$\begin{array}{ccc} Z' & \hookrightarrow & \mathbb{V}(V) \times \mathbb{P}(\vec{W}) \\ \downarrow & & \downarrow \\ Z & \hookrightarrow & \mathbb{V}(V \otimes W^*) \times \mathbb{P}(\vec{W}). \end{array}$$

The dimension of  $Z$  is

$$\dim \mathbb{P}(\vec{W}) + \dim_k V \cdot \operatorname{rk} \mathcal{Q} = \dim \mathbb{P}(\vec{W}) + \dim_k T.$$

By Corollary 4.4 the dimension of  $Z'$  is less than or equal to

$$\dim \mathbb{P}(\vec{W}) = \dim Z - \dim_k T.$$

Let  $Y'$  be the pullback in the diagram

$$\begin{array}{ccc} Y' & \hookrightarrow & \mathbb{V}(V) \\ \downarrow & & \downarrow \\ Y & \hookrightarrow & \mathbb{V}(V \otimes W^*). \end{array}$$

Since the image of  $Z$  is  $Y$ , the image of  $Z'$  is  $Y'$ . This gives

$$\dim Y' \leq \dim Z' \leq \dim Z - \dim_k T = \dim Y - \dim_k T.$$

The complex  $F^\bullet(V; \vec{W})$  is a resolution of a Cohen-Macaulay module  $M$  supported on  $Y$ . The module  $M' = M \otimes_{S(V \otimes W^*)} S(V)$  where  $S(V) = S(V \otimes W^*)/(T)$  is supported on  $Y'$  and so

$$\dim M' \leq \dim Y' \leq \dim Y - \dim_k T = \dim M - \dim_k T.$$

Since  $M' = M/(T \cdot M)$ , a basis for  $T$  must form a regular sequence, and so  $M'$  is a Cohen-Macaulay module with resolution given by  $F^\bullet$ . Therefore  $F^\bullet$  becomes a pure resolution of a Cohen-Macaulay where the terms with generators of degree  $d \in A$  are

$$(20) \quad \bigwedge^d V \bigotimes (\bigotimes_{d \leq u_i} S^{u_i-d}(W_i)) \bigotimes (\bigotimes_{d \geq u_i+w_i} \tilde{D}^{d-u_i+w_i}(W_i)) \bigotimes S(V).$$

The basis  $x_1, \dots, x_n$  of  $V$  induces a maximal torus  $D$  of  $GL(V)$ , the diagonal matrices. The quotient map (19) is equivariant for the torus action where  $t = (t_1, \dots, t_n) \in D$  acts on  $w = \sum x_i \otimes w_i^*$  as  $t.w = \sum (t_i x_i) \otimes w_i^*$ . Thus the complex above is equivariant for the torus action and so is  $\mathbb{Z}^n$ -graded. The action on the term (20) in the complex is given by the natural actions on  $\wedge^d V$  and  $S(V)$  and the trivial action on the rest of the tensor factors. Hence the multidegrees of the generators of the terms above are of squarefree degree, and so the resolution is squarefree.  $\square$



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